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A new measure of network efficiency

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ABSTRACT

We address the issue of the dynamical origin of scale-free link distributions. We study a two-dimensional lattice of cooperatively interacting units. Although the units interact only with the four nearest neighbors, a sufficiently large cooperation strength generates dynamically a scale-free network with the power law index ν approaching 1. We explain this result by using a new definition of network efficiency determined by the Euclidean distance between correlated units. According to this definition the link distribution favoring long-range connections makes efficiency increase. We embed an *ad hoc* scale-free network with power index $\nu \geq 1$ into a Euclidean two-dimensional space and show that the network efficiency becomes maximal as ν approaches 1. We therefore conclude that $\nu = 1$ emerging from the cooperative interaction of units may be a consequence of the principle of network maximal efficiency.

1. Introduction

The work of Albert and Barabási (AB) [1] prompted interest in scale-free networks, namely networks with the inverse power law distribution of links

$$p(k) \propto \frac{1}{k^{\nu}}, \tag{1}$$

with k denoting the number of links per node. The AB model assumed the preferential attachment of the Yule process and gave rise to the power-law index $\nu = 3$. The ubiquity of these networks has led to the research into the dynamical origin of the scale-free property. Fraiman et al. [2] has recently studied the two-dimensional Ising model at criticality and using a suitable threshold ρ^+ established a link between the nodes with a dynamical correlation exceeding that threshold. The interesting result of that procedure was the emergence of the scale-free distribution density of Eq. (1) with a power index ν that, with the proper choice of average connectivity $\langle k \rangle$, gives values close to $\nu = 2$ [3]. It is remarkable that the same procedure, applied to the brain, yields results very close to those afforded by the Ising model [2].

Like the work of Ref. [2], we begin with a regular two-dimensional network, but then take an entirely different approach. Rather than using the Ising model, we adopt the Decision Making (DM) model of Refs. [4,5] for the creation of the dynamically induced network. The key result is that while we also find the emergence of a scale-free network at criticality, our network has an exponent ν close to 1.

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This finding suggests that the cooperative interaction between the units in the network generated from the DM model should make the network very efficient. However, we find herein that the conventional measures of network efficiency [6] are relatively insensitive to $\nu < 2$. The second result of this paper is a new measure of network efficiency, the perception length, that overcomes this limitation, and shows that network efficiency increases as ν approaches 1.

The outline of this paper is as follows. We devote Section 2 to proving the dynamical emergence of the scale-free network with $\nu = 1$. Section 3 shows that ν approaching 1 maximizes the perception length while realizing a statistical condition equivalent to the dynamically generated network. We devote Section 4 to concluding remarks.

2. Dynamically induced scale-free network with $\nu = 1$

The question of whether the network emerging from dynamical correlations is scale-free is not trivial. For this paper, we adopt the Decision Making (DM) model of Refs. [4,5]. Although the DM model yields a phase transition very similar to that of the conventional Ising model [7], the DM model does not have a Hamiltonian foundation. Furthermore, while the Ising model rests on the action of a thermal bath at a finite temperature, and the single spins in isolation and with no thermal driving would be dynamically frozen, the units of the DM model, in isolation, are driven by Poisson dynamics. This difference may lead to a dynamically created scale-free network different from that of Ref. [2].

The DM model considers N discrete variables located at the nodes of a two-dimensional square lattice. Each unit s_i is a stochastic oscillator and can be found in either of two states, "yes" $(+1)$ or "no" (-1). The cooperation among the units is realized by setting the transition rates between two states to the time-dependent form:

$$g_{12}(t) = g e^{K \frac{M_2(t)-M_1(t)}{M}} \tag{2}$$

and

$$g_{21}(t) = g e^{-K \frac{M_2(t)-M_1(t)}{M}} \tag{3}$$

where M denotes the total number of nearest neighbors, which in the case of a two-dimensional lattice results in $M = 4$. $M_1(t)$ and $M_2(t)$ are the nearest neighbors who are making the decision "yes" and "no", respectively. The single unit in isolation, $K = 0$, fluctuates between "yes" and "no", with the rate g. When coupling constant $K > 0$, a unit in the state "yes" ("no") makes a transition to the state "no" ("yes") faster or slower according to whether $M_2 > M_1$ $(M_1 > M_2)$ or $M_2 < M_1$ $(M_1 < M_2)$, respectively. The quantity K_C is the critical value of the control parameter K, at which point a phase-transition to a global majority state occurs. It can be shown that for a lattice of infinite size $K_C = 2\ln(1+\sqrt{2}) \approx 1.7627$.

All simulations are implemented on a lattice of $N = 100 \times 100$ nodes with periodic boundary conditions. In a single time step, a run over the entire lattice is performed and for every node an appropriate transition rate (g_{12} or g_{21}) is calculated according to which a node is given possibility to change its state. Under those conditions, setting parameter $g = 0.01$, the phase transition to the global majority case occurs at $K_C \approx 1.70$. To study the dynamically induced network topology, we considered the DM model with this critical value of coupling constant. After an initial 10^6 time steps, we record 2000 lattice configurations, obtaining the dynamics of each node $\{s_i(t)\}$. In the next step, we evaluate the linear correlation coefficient between the i-th and the j-th node:

$$r(i,j) = \frac{\langle s_i(t)s_j(t)\rangle - \langle s_i(t)\rangle\langle s_j(t)\rangle}{\sqrt{\langle s_i^2(t)\rangle - \langle s_i(t)\rangle^2}\sqrt{\langle s_j^2(t)\rangle - \langle s_j(t)\rangle^2}}, \tag{4}$$

where $\langle \cdots \rangle$ stands for the time average. Nodes i and j of the dynamically induced network are connected by a link when the correlation between nodes i and j of the two-dimensional lattice is greater than a given positive threshold ρ^+.

We investigated a wide range of positive thresholds $\rho^+ = [0.10 : 0.90]$. As expected, correlation networks constructed with increasing value of the threshold ρ^+ are characterized by a decreasing number of links, which results in significant changes in the corresponding degree distribution. Networks obtained with low values of the threshold ρ^+ result in narrow, peaked degree distributions and relatively high values of the mean degree $\langle k\rangle$. Topologies resulting from very strong correlations, however, are characterized by a distribution of links that has a mean close to the minimum value of $\langle k\rangle = 1$ and drops off very rapidly. Interestingly, in between those conditions there exists a correlation threshold that creates a dynamically induced network with a power-law distribution of degrees that is shown in Fig. 1. The connectivity of a network obtained under this particular condition is inspected and only nodes that are part of a giant component are considered to be a part of new topology. The resulting network consists of 6740 nodes with a mean degree of $\langle k\rangle = 38$. For a correlation threshold $\rho^+ = 0.61$, we find that the distribution of links follows an inverse power law with the scaling parameter $\nu = 1.20 \pm 0.23$ ($R^2 = 0.9556$).

How can the dynamical origin of this extremal condition be explained? From an intuitive point of view there exists a connection between the scale-free distribution of links and Zipf's law [8,9]. In fact, if we rank the nodes moving from the richest to the poorest, according to Zipf [10] we should obtain the relation between the connectivity k and rank r:

$$k(r) = \frac{C}{r^{\alpha}}, \tag{5}$$

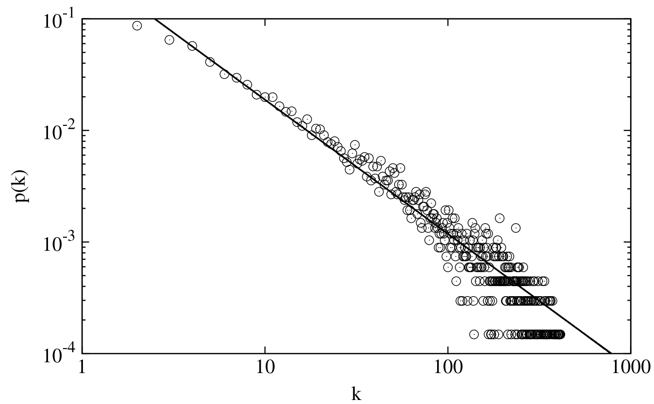

Fig. 1. Degree distribution $p(k)$ for correlation network obtained with a positive threshold $\rho^+ = 0.61$. The line follows estimated power-law relation with the exponent of $\nu = 1.20$.

with the power-law index α being very close to 1. If we select randomly a given rank r with probability $\pi(r)\mathrm{d}r$ of being in the interval $[r, r + \mathrm{d}r]$, this should be equal to the probability $p(k)\mathrm{d}k$, of having k links in the interval $[k, k + \mathrm{d}k]$,

$$p(k)\mathrm{d}k = \pi(r)\mathrm{d}r. \tag{6}$$

Assuming that $\pi(r)$ is constant, we obtain after some algebra

$$p(k) \propto \frac{1}{k^{1+\frac{1}{\alpha}}}, \tag{7}$$

and thus the two power-law indices are related by

$$\nu = 1 + \frac{1}{\alpha}. \tag{8}$$

This leads us to conclude how an efficient complex network is organized if Zipf's law applies. For example, this suggests that the intelligence of human brain should be characterized by $\nu = 2$. The recent research work of Fraiman et al. [2] seems to confirm this prediction. The authors of Ref. [2] studied the two-dimensional Ising model at criticality and using a suitable threshold ρ^+ established a link between the nodes with a dynamical correlation exceeding that threshold. The interesting result of that procedure was the emergence of the scale-free distribution density of Eq. (1) with a power index ν that, with the proper choice of average connectivity $\langle k \rangle$, gives values close to $\nu = 2$ [3]. It is remarkable that the same procedure, applied to the brain, yields results very close to those afforded by the Ising model [2].

It is also interesting to notice that Boettcher and Percus [11] applied the method of extremal optimization to a Ising-like model to evaluate the rank distribution of Eq. (5) and found that when the number of units tend to infinity the rank distribution tends to $\alpha = 1$. This result implies on the basis of the earlier remarks that the cooperation of units in an Ising-like model is expected to confirm Zipf's prediction and consequently $\nu = 2$. For results confirming this prediction, see also Refs. [12,13].

We have found, therefore, that the DM model generates a scale-free network at criticality, but with ν approaching 1 which violates Zipf's condition of $\alpha = 1$. The emergence of $\alpha \gg 1$ corresponds to the number of links k dropping very quickly when we move from $r = 1$ to subsequent ranks.

3. Embedding scale-free networks in a two-dimensional regular lattice

The efficiency of a network has been mainly studied on the basis of topological length L [6]. The topological distance between two nodes is the minimum number of steps needed to move from one node to another. The quantity L is the corresponding mean value of the minimum topological distance between nodes. As a consequence, the emergence of scale-free networks can be explained by making the conjecture that real networks evolve in time so as to realize maximal efficiency. In the case of the scale-free network this efficiency is measured by the minimal value of L.

In 2003, Cohen and Havlin [14] determined that scale-free networks are very efficient. They calculated that networks with N nodes have a topological length L given by

$$L \sim \ln(\ln N), \tag{9}$$

for a power-law index $\nu < 3$, which is smaller than in the AB model.

As a consequence of Eq. (9), the topological length L remains small even if N is very large. This relation between L and N suggests that the complex networks with $\nu < 3$ are very efficient.

The region $\nu < 2$ has not been studied by many authors. As an interesting example of earlier work in this region we refer to the work of Refs. [15,16]. In both cases the authors adopt a kind of generalization of the perspective of Albert and

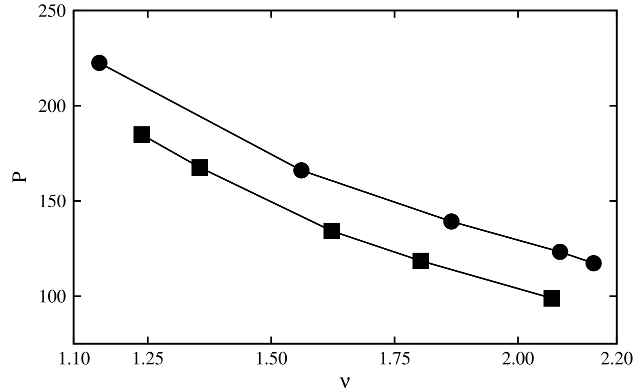

Fig. 2. Perception length P as a function of ν for *ad hoc* scale-free networks. The circles represent networks with constant $\langle k\rangle = 4.95$; the squares represent networks with constant $N = 1313$.

Barabási [1], namely, they grow a network with special prescriptions allowing them to overcome the limiting condition $\nu > 3$ and yielding $\nu < 2$.

More recently there has been some interest in the efficiency of hierarchical networks with $\nu < 2$ [17] and $\nu \sim 1$ [18,19]. These networks are characterized by very small topological lengths. In the case of Ref. [19], for $N \to \infty$ the authors predict the value $L = 2$. Our numerical calculations show that L is insensitive to ν in the range $1 < \nu < 2$. Thus, there are good reasons to believe that the prediction of Eq. (9) applies with no significant dependence on ν.

There are also good reasons to believe that the dynamic source of scale-free networks has a probabilistic rather than a deterministic origin, as we shall demonstrate. The work of Fraiman et al. [2], although fitting Zipf's condition, allows us to define the concept of perception length, which is then used to shed light onto the condition $\nu = 1$. The nodes of a two-dimensional regular lattice, where each unit interacts with only its four nearest neighbors, correspond to a network whose topological efficiency is very low. However, if we let the units cooperatively interact, and we set a link between two nodes, then when their dynamical correlation is large enough, the resulting dynamical network may have a much lower L. We record the Euclidean distance between each pair of correlated nodes, and we define the network efficiency as the corresponding mean value. More precisely, we define the network efficiency by means of the quantity

$$P = \frac{1}{N}\sum_{1}^{N} \lambda_i, \tag{10}$$

where

$$\lambda_i = \sum_{j=1}^{k_i} d_{ij} \tag{11}$$

and d_{ij} is the Euclidean distance between node i and its nearest neighbor j. We refer to P as the perception length of the system, this being the measure of the network efficiency replacing $1/L$. For numerical analysis, the perception length P is calculated as follows.

1. Create a 2-dimensional square lattice, where a is the length (in units of the lattice) of the side of the square, and it is necessary that $\sqrt{N} < a$.
2. Embed the network on the lattice by randomly assigning the network nodes to lattice points.
3. Using periodic boundary conditions for the lattice, calculate the Euclidean length of each link.
4. These lengths are then used to calculate P in accordance with Eq. (10).

This procedure was then applied to scale-free networks generated according to the method of Catanzaro et al. [20] with $\langle k\rangle = 4.95$ and values of ν ranging from 1 to 2. The nodes were embedded randomly in a square lattice of size just sufficient to contain the network with the largest number of nodes. The perception lengths were calculated, and the results show that over this range the perception length increases as ν approaches 1 (Fig. 2).

We now compare the sensitivity of the conventional network efficiency (defined as the reciprocal of the network length *1/L*) and the perception length P to the variation in ν. These results are shown in Fig. 3. Note that the perception length P and $1/L$ have been normalized to allow the presentation in a convenient scale. These results show that the perception length, as a measure of network efficiency, is more sensitive to changes in ν than the conventional $1/L$.

Applying this approach to the DM network discussed in Section 2 above yields a mean perception length = 1382. To provide a basis for comparison, we calculated the perception length for a comparably sized *ad hoc* scale-free network, also prepared in accordance with the algorithm in Ref. [20]. The results are compared in Table 1, which shows a very good agreement.

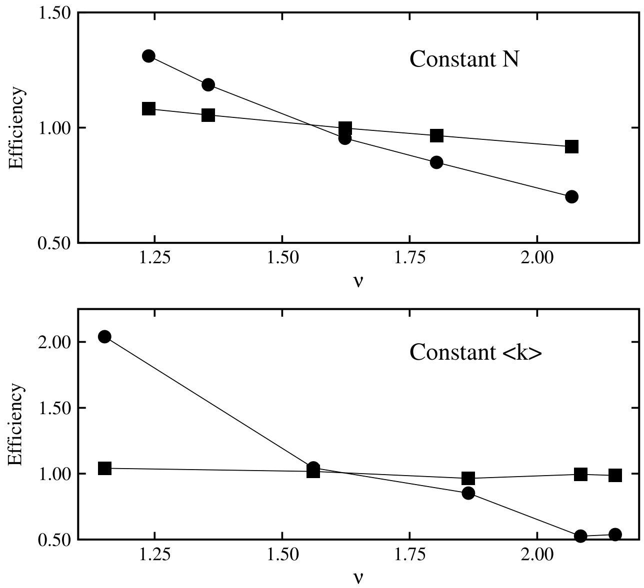

Fig. 3. Network efficiency as a function of ν. In both figures, the circles represent the normalized perception length, and the squares represent the normalized $1/L$. Error bars are not shown; standard deviations for the normalized perception lengths were very small (of the order of 0.005).

Table 1
Comparison of perception lengths for the DM network and a comparable *ad hoc* scale-free network.

	DM network	*Ad Hoc* network
Degree distribution exponent, ν	1.20	1.19
Number of nodes, N	6740	6744
Number of links, k	128,448	128,413
Perception length, P	1382	1373

4. Concluding remarks

The main result of this paper is the proof that the perception length P defined by Eq. (10) is more sensitive to ν in the range $1 < \nu < 2$, and may therefore be a more useful measure of network efficiency than the topological length L [6]. This new definition is based on the assumption that the scale-free networks, even those derived using deterministic and hierarchical arguments [17–19], are compatible with the dynamical derivation. The apparent conflict between the two kinds of network is explained by noticing that the correlation between different nodes evaluated according to the numerical prescription of Section 2 rests on time windows of finite size and moving these windows to different time regions generate different networks with the same hierarchical topology. An alternative but probably more attractive way to explain this is that the leadership moves in time from some nodes to others. The nodes of the two-dimensional regular lattice that we used for the dynamical derivation of the free-scale distribution of Eq. (1) are totally equivalent, and the rank differences between the richest and the poorest nodes are temporary and in the long time run the same high rank is shared by all the nodes.

There are some questionable aspects of the dynamical derivation of $\nu = 1$, realized in Section 2. First of all this result depends on an arbitrarily selected threshold ρ^+. In addition, the power index ν derived from the data in Fig. 1 is affected by the large fluctuations of the distribution density $p(k)$ for large values of k, thereby explaining why we derive $\nu = 1$ rather than $\nu = 2$, as in the work of Ref. [3]. However, on one hand, this observation led us to the discovery of the new measure of network efficiency of Eq. (10) and, on the other hand, as already mentioned in Section 1, the DM model is not quite equivalent to the Ising model.

Furthermore, the interesting fact that $\alpha = 1$ (Zipf's law) is related to $\nu = 2$ is not totally convincing. The authors of Ref. [9] established a connection between topological and dynamical properties by running a random walker on a scale-free network with the power index ν. They found that the waiting time distribution density for returns of the walker to a given node are described by an inverse power law with power index $\mu = 3-\nu$. Using a generalized central limit theorem, they also proved that the same index μ also describes the inverse power law form of the probability density of relative frequencies $p(f) \sim f^{-\mu}$, so that, in analogy with Eq. (8), a rank-frequency law $f \sim r^{-\alpha}$ yields $\mu = 1 + 1/\alpha$, hence $\alpha = 1/(2 - \nu)$, namely $\alpha = 1$ for $\nu = 1$. It is interesting to note that $\nu = 1$ yields $\mu = 2$, and that the value $\mu \approx 2$ emerges from the recent neurophysiological work of Refs. [21,22].

Thus, in addition to the new efficiency measure of Eq. (10), which is the unquestionable result of this paper, we are also led to make the plausible conjecture that the DM model may be an adequate tool to study the emergence of intelligence

in the sense pointed out by Couzin [23] and Cavagna et al. [24]. According to these authors the flocks of birds, as well as the brain [25,26], are systems at criticality, with a correlation length as large as the size of these systems. We are also led to make the conjecture that the dynamical generation of a scale-free network with $\nu = 1$ may be the manifestation of the principle of network maximal efficiency: a network of cooperatively interacting units evolves naturally so as to maximize P, the new measure of network efficiency.

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